Section 3.2

In general n-th order linear differential equation is of the form. $P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x)$ —(1)

where the coefficient functions $P_i(x)$ and $F(x)$ are cts on some open interval I. Further, let us assume that $P_0(x) \neq 0$ at each point of I. then dividing eq^n (1) by $P_0(x)$ we obtain alternative form of nth-order linear differential eq^n as –

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x) \quad \text{—(2)}$$

The associated homogeneous eq^n is –

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \quad \text{—(3)}.$$

Theorem 1. Principle of Superposition for homogeneous Equations

Let $y_1, y_2, \ldots y_n$ be n solutions of the homogeneous linear equation in (3) on the interval I. If $c_1, c_2, \ldots c_n$ are constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \quad (4)$$ is also a sol^n of eq^n (3) on I

Ex:- Consider the differential eq^n $y^{(3)} + 3y'' + 4y' + 12y = 0$ —(a)

Here $y_1 = e^{-3x}$, $y_2 = \cos 2x$ and $y_3 = \sin 2x$ are all solutions of the homogeneous third order differential equation given by (a) on the entire real line. Thus by principle of superposition for homogeneous eq^{n}s. any linear combination of these solutions such as $y(x) = -3y_1 + 3y_2 - 2y_3$ is also a solution on the entire real line. Conversely, we can say every solution of the differential eq^n is a linear combination of the three particular solutions y_1, y_2 and y_3. Thus, a general solution is given by $y = c_1e^{-3x} + c_2\cos 2x + c_3\sin 2x$.

Existence and Uniqueness of Solution

As we require two initial conditions to find a particular solution of second order linear differential eq". Similarly we require n initial conditions to solve n^{th} order linear differential equation.

Theorem 2:- Existence and Uniqueness for linear Equations

Suppose that the functions $p_1, p_2, \ldots p_n$ and f are continuous on the 'open interval I containing the point a. Then, given n numbers $b_0, b_1, \ldots, b_{n-1}$ the n^{th} order linear eq^n in (2).

i.e. $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x)$

has a unique solution on the entire interval I that satisfies the initial condition.

$y(a) = b_0, \; y'(a) = b_1, \; y''(a) = b_2 \; \ldots \; y^{(n-1)}(a) = b_{n-1}$ — (5)

Eq^n(2) and conditions in (5) together constitute an nth order initial value problem.

Ex. The solution of the differential eq^n $y^{(3)} + 3y'' + 4y' + 12y = 0$ for initial values $y(0) = 0$, $y'(0) = 5$ and $y''(0) = -39$ is $y = -3e^{-3x} + 3\cos 2x - 2\sin 2x$. Theorem 2 implies that there is no other solution with these same initial values.

Note :- The trivial solution $y(x) \equiv 0$ is the only solution of the homogeneous equation

$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0$

that satisfies the trivial initial conditions

$y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0.$

Def^n. Linear Dependence of functions

The no functions $f_1, f_2, \dots f_n$ are said to be linearly dependent on the interval I provided that $\exists$ constants $c_1, c_2, \dots c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \text{ on } I. \quad \text{———} (6)$$

i.e. $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for all $x \in I$.

Note :- If not all the coefficients in eq^n (6) are zero, then clearly we can solve for at least one of the functions as a linear combination of the others, and conversely. Thus the functions $f_1, f_2, \dots f_n$ are L.D. Iff at least one of them is a linear combination of the others.

Ex. Consider the functions

$f_1(x) = \sin 2x$, $f_2(x) = \sin x \cos x$ and $f_3(x) = e^x$.

We have $\sin 2x = 2 \sin x \cos x$

or $f_1(x) = 2 f_2(x)$

$\Rightarrow (1) f_1(x) + (-2) f_2(x) + 0 f_3(x) = 0.$

On comparing it with $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ we obtain c_1 and $c_2 \neq 0$ and hence $f_1(x)$, $f_2(x)$ and $f_3(x)$ are linearly dependent on the real line.

Defn. Linearly Independent functions

Then n functions $f_1, f_2, - f_n$ are linearly independent on I provided that the identity

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \quad \text{---} (5)$$

holds on I only in the trivial case where $c_1 = c_2 = \cdots = c_n = 0$, i.e. no nontrivial linear combination of these function vanishes on I.

Note :- The functions $f_1, f_2, -- f_n$ are LI if none of them can be written as a linear combination of the others.

Wronskian.

Wronskian of n function $f_1, f_2, - f_n$ which are $(n-1)$ times differentiable is denoted by $W(n)$ or $W(f_1, f_2, - f_n)$ and is defined as

$$W(n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \quad \text{---} (7)$$

Wronskian of n linearly dependent functions

The Wronskian of n linearly dependent functions $f_1, f_2, -- f_n$ is identically zero.

Proof :- Assume that $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ (6) holds on the interval I for some choice of the constants $c_1, c_2, - c_n$ not all zero. Differentiating eqn (6) $(n-1)$ times we obtain n equations as—

$$\left.\begin{aligned} c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) &= 0 \\ c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) &= 0 \\ &\vdots \\ c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) &= 0 \end{aligned}\right] \quad (8).$$

which holds for all x in I. We know that a system of n linear homogeneous eq^n in n unknowns has a nontrivial solution iff. the determinant of coefficients vanishes. In eq^n (8) the ~~coefficient~~ unknowns are the constants $c_1, c_2 - c_n$ and the determinant of the coefficients is simply the wronskian $W(f_1, f_2 - - f_n)$. evaluated at point x in I. Since c_i are not all zero, it follow that $W(x) \equiv 0$.

\# To show that the functions $f_1, f_2, - f_n$ are LD on the interval I, it is sufficient to show that their wronskian is nonzero at just one point of I.

Ex: Show that the functions $y_1(x) = e^{-3x}$, $y_2(x) = \cos 2x$ and $y_3(x) = \sin 2x$ are linearly independent.

Sol.

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= e^{-3x}\begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} + 3e^{-3x}\begin{vmatrix} \cos 2x & \sin 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} + 9e^{-3x}\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 8e^{-3x} + 0 + 18e^{-3x} = 26e^{-3x} \neq 0.$$

$\because$ $W \neq 0$ everywhere, thus y_1, y_2, y_3 are Linearly independent on ~~the~~ any open interval.

Q11. Show that the three solutions

$y_1(x) = x$, $y_2(x) = x\ln x$ and $y_3(x) = x^2$

of the third-order D.E.

$$x^3 y^{(3)} - x^2 y'' + 2xy' - 2y = 0 \quad \text{---(1)}$$

are linearly independent on the open interval $x > 0$.

Then find a particular solution of eq^n (1) that satisfies the initial conditions $y(1) = 3$, $y'(1) = 2$, $y''(1) = 1$ (1a)

Ans. $\because x > 0$, we can divide eq^n (1) by x^3 to obtain a homogeneous linear eq^n of the standard form

$$y^{(3)} + p_1(x)y'' + p_2(x)y' + p_3 y = 0.$$

Now
$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & \ln x + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$= x\left[2\ln x + \cancel{2} - \cancel{2}\right] - x\ln x\left[2\right] + x^2\left[\frac{1}{x} - 0\right]$$

$$= 2x\ln x - 2x\ln x + x = x \neq 0.$$

Hence the three solutions are L.I. on the interval $x>0$.

Let $y = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$ be the required sol^n.

Then $y = c_1 x + c_2 x\ln x + c_3 x^2$

$$y' = c_1 + c_2(1 + \ln x) + 2c_3 x$$

$$y'' = 0 + \frac{c_2}{x} + 2c_3.$$

Now $y(1) = 3 \Rightarrow c_1 + c_3 = 3$ ---(a)

$y'(1) = 2 \Rightarrow c_1 + c_2 + 2c_3 = 2$ ---(b)

$y''(1) = 1 \Rightarrow c_2 + 2c_3 = 1$ ---(c)

From (b) & (c) we get $c_1 = 1$ & from (a) $c_3 = 2$, (c) $\Rightarrow$ $c_2 = 1 - 4 = -3$. Hence the reqd sol^n $\underline{y = x + (-3)x\ln x + 2x^2}$.

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Theorem 3 Wronskian of Solutions

Suppose that $y_1, y_2, \ldots y_n$ are n solutions of the homogeneous nth-order linear equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = 0 \quad \text{---} (3)$$

on an open interval I where each p_i is cts.

Let $W = W(y_1, y_2, \ldots y_n)$.

(a) If $y_1, y_2, \ldots y_n$ are linearly dependent, then $W \equiv 0$ on I.

(b) If $y_1, y_2, \ldots y_n$ are linearly independent then $W \neq 0$ at each point of I.

Proof:- Proof of part (a) we have already done (refer page nos 4 & 5).

To prove part (b), it is sufficient to assume that $W(a) = 0$ at some point a of I and show this implies that the solutions $y_1, y_2, \ldots y_n$ are linearly dependent. But $W(a)$ is simply the determinant of coefficients of the system of n homogeneous linear equations.

$$\left.\begin{aligned}
&c_1 y_1(a) + c_2 y_2(a) + \cdots + c_n y_n(a) = 0 \\
&c_1 y_1'(a) + c_2 y_2'(a) + \cdots + c_n y_n'(a) = 0 \\
&\quad\vdots \\
&c_1 y_1^{(n-1)}(a) + c_2 y_2^{(n-1)}(a) + \cdots + c_n y_n^{(n-1)}(a) = 0
\end{aligned}\right] \text{---} (11)$$

in the n unknowns $c_1, c_2, \ldots c_n$. Because $W(a) = 0$ implies that the c_i's in (11) have a nontrivial sol. That is the numbers $c_1, c_2, \ldots c_n$ are not all zero. Thus, $\exists$ a particular solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \quad \text{---} (12) \text{ of eq}^n (3)$$

The eq^n in (11) then imply that

$$Y(a) = c_1 y_1(a) + c_2 y_2(a) + \cdots + c_n y_n(a) = 0$$

$$Y'(a) = c_1 y_1'(a) + c_2 y_2'(a) + \cdots + c_n y_n'(a) = 0$$

$$\vdots$$

$$Y^{(n-1)}(a) = c_1 y_1^{(n-1)}(a) + c_2 y_2^{(n-1)}(a) + \cdots + c_n y_n^{(n-1)}(a) = 0$$

i.e. Y satisfies the trivial initial ~~values~~ conditions

$$Y(a) = Y'(a) = \cdots = Y^{(n-1)}(a) = 0.$$

Thus $Y(x) \equiv 0 \ \forall x$ in I. as we have from uniqueness and existence theorem the solution should be unique

thus, $Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0 \ \forall x$ in I.

$\because$ $c_1, c_2, \ldots c_n$ are not all zero. Thus,

$y_1, y_2, \ldots y_n$ are Linearly dependent.

This completes the proof of the theorem.

Theorem 4 **General Solutions of Homogeneous Equations**

Let $y_1, y_2, \ldots y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x) y^{(n-1)} + p_2(x) y^{(n-2)} + \cdots + p_{n-1}(x) y' + p_n(x) y = 0 \quad (3)$$

on an open interval $I \subset R$ where p_i are cts. If Y is any solution whatsoever of eq^n (3) then $\exists$ numbers $c_1, c_2, \ldots c_n$ such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \quad \forall x \in I.$$

Proof :- Try yourself as we did in Sec 3.1 Theorem 4.

Note :- Every solution of a homogeneous n^{th}-order linear differential equation is a linear combination of any n given linearly independent solutions i.e. $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$. Such a linear combination is called a general solution of the differential equation.

Linear Second-order Equations with Constant Coefficients

Suppose a homogeneous second order, linear differential equation with constant coefficients is given by –

$$ay'' + by' + cy = 0 \qquad \text{------} \quad (1)$$

where a, b and c are some constants.

Let $y = e^{rx}$ is solution of eq^n (1). Then it must satisfy it.

So, $a[r^2e^{rx}] + b[re^{rx}] + ce^{rx} = 0$

$\Rightarrow e^{rx}[ar^2 + br + c] = 0$

$\Rightarrow ar^2 + br + c = 0$ —(2) $\{\because e^{rx} \neq 0$ which is solution of the given differential eq^n$\}$

which is a quadratic eq^n in r. This quadratic equation is called the $\boxed{\text{characteristic / auxiliary eq}^n}$. of the homogeneous ~~So we get two roots for r.~~ linear differential equation

$ay'' + by' + cy = 0$.

Eq (2) has two roots. Thus, we have three possibilities. First, the roots are real and distinct. Second the roots are real and equal. Third the roots are complex.

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Case I: Roots are real and distinct.

Theorem 5. Distinct Real roots

If the roots r_1 and r_2 of the characteristic eqn $ar^2+br+c=0$ are real and distinct, then $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the general soln of eqn (1) where $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are the two solutions. Also, we can check that y_1 and y_2 are LI solutions.

Ex. Find the general solution of

$$2y'' - 7y' + 3y = 0.$$

Soln Here the auxiliary/ characteristic eqn is

$$2r^2 - 7r + 3 = 0.$$

$\Rightarrow$ $2r^2 - 6r - r + 3 = 0$

$\Rightarrow$ $2r(r-3) - 1(r-3) = 0$

$\Rightarrow$ $(2r-1)(r-3) = 0$

$\Rightarrow$ $r = \frac{1}{2}$ and 3

Thus $r_1 = \frac{1}{2}$ and $r_2 = 3$ which are real and distinct.

So, $y_1 = e^{r_1 x} = e^{x/2}$ and $y_2 = e^{r_2 x} = e^{3x}$.

Hence the general solution is -

$$y = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 e^{x/2} + c_2 e^{3x}$$ where c_1 and c_2 are arbitrary constants.

Ex. Solve the differential eqⁿ $y'' + 2y' = 0$.

Sol. The characteristics eqⁿ is $r^2 + 2r = 0$

$\Rightarrow r(r+2) = 0$

$\Rightarrow r = 0, r = -2$.

Hence $y_1 = e^{0x} = 1$, $y_2 = e^{-2x}$.

So the general solution is $y = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 + c_2 e^{-2x}$ where c_1 and c_2 are

arbitrary constants.

Case II. If roots are real and repeated.

In that case one root r_1 is repeated. So one of the solution is $y_1 = e^{r_1 x}$ and the second L.I. solution is $y_2 = x e^{r_1 x}$.

Theorem 6. Repeated Roots.

If the characteristics eqⁿ in (1) has equal roots $r_1 = r_2$. then $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

$= (c_1 + c_2 x) e^{r_1 x}$ is a general

solution of eqⁿ (1).

Ex. Solve the IVP $y''+2y'+y=0$.
$y(0)=5,\ y'(0)=-3$.

Sol Here the characteristic eqn is
$r^2+2r+1=0$
$\Rightarrow (r+1)^2=0 \Rightarrow r=-1,-1$.

$\therefore\ y=(c_1+c_2x)e^{-x}$ is the general soln where c_1 and c_2 are some arbitrary constants.

$\because\ y(0)=5 \Rightarrow (c_1+c_2.0)e^{-0}=5$
$\Rightarrow\ c_1=5$.

Also, $y'=-(c_1+c_2x)e^{-x}+c_2e^{-x}$
$y'(0)=\ -c_1+c_2=-3$
$\Rightarrow\ c_2=-3+c_1=-3+5=2$.

Thus, $y=(5+2x)e^{-x}$ is the desired particular solution of the given IVP.